1+3 covariant dynamics of scalar perturbations in braneworlds

Bernard Leong,[1,*] Peter Dunsby,[2,†] Anthony Challinor,[1,‡] and Anthony Lasenby[1,§]

[1]*Astrophysics Group, Cavendish Laboratory, Madingley Road, Cambridge CB3 OHE, UK*

[2]*Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, South Africa*

(Dated: February 7, 2008)

We discuss the dynamics of linear, scalar perturbations in an almost Friedmann-Robertson-Walker braneworld cosmology of Randall-Sundrum type II using the 1+3 covariant approach. We derive a complete set of frame-independent equations for the total matter variables, and a partial set of equations for the non-local variables which arise from the projection of the Weyl tensor in the bulk. The latter equations are incomplete since there is no propagation equation for the non-local anisotropic stress. We supplement the equations for the total matter variables with equations for the independent constituents in a cold dark matter cosmology, and provide solutions in the high and low-energy radiation-dominated phase under the assumption that the non-local anisotropic stress vanishes. These solutions reveal the existence of new modes arising from the two additional non-local degrees of freedom. Our solutions should prove useful in setting up initial conditions for numerical codes aimed at exploring the effect of braneworld corrections on the cosmic microwave background (CMB) power spectrum. As a first step in this direction, we derive the covariant form of the line of sight solution for the CMB temperature anisotropies in braneworld cosmologies, and discuss possible mechanisms by which braneworld effects may remain in the low-energy universe.

PACS numbers: 04.50.+h, 98.80.Cq

I. INTRODUCTION

It is understood that Einstein's theory of general relativity is an effective theory in the low-energy limit of a more general theory. Recent developments in theoretical physics, particularly in string theory or M-theory, have led to the idea that gravity is a higher-dimensional theory which would become effectively four-dimensional at lower energies.

Braneworlds, which were inspired by string and M-theory, provide simple, yet plausible, models of how the extra dimensions might affect the four-dimensional world we inhabit. There is the exciting possibility that these extra dimensions might reveal themselves through specific cosmological signatures that survive the transition to the low-energy universe. It has been suggested that in the context of braneworld models the fields that govern the basic interactions in the standard model of particle physics are confined to a 3-brane, while the gravitational field can propagate in $3+d$ dimensions (the *bulk*). It is not necessarily true that the extra dimensions are required to be small or even compact. It was shown recently by Randall and Sundrum [1] for the case $d=1$, that gravity could be localized to a single 3-brane even when the fifth dimension was infinite. As a result, the Newtonian potential is recovered on large scales, but with a leading-order correction on small scales:

$$V(r) = -\frac{GM}{r}\left(1+\frac{2l^2}{3r^2}\right) \,, \tag{I.1}$$

where the 5-dimensional cosmological constant $\tilde{\Lambda} \propto -l^{-2}$. As a result, general relativity is recovered in 4 dimensions in the static weak-field limit, with a first-order correction which is believed to be constrained by sub-millimeter experiments at the TeV level [1, 2].

The cosmic microwave background (CMB) currently occupies a central role in modern cosmology. It is the cleanest cosmological observable, providing us with a unique record of conditions along our past light cone back to the epoch of decoupling when the mean free path to Thomson scattering rose suddenly due to hydrogen recombination. Present (e.g. BOOMERANG [3]) and MAXIMA [4]) and future (e.g. MAP and PLANCK) data on the CMB anisotropies and large-scale structure provide extensive information on the spectrum and evolution of cosmological perturbations

*Electronic address: cwbl2@mrao.cam.ac.uk
†Electronic address: peter@vishnu.mth.uct.ac.za
‡Electronic address: A.D.Challinor@mrao.cam.ac.uk
§Electronic address: A.N.Lasenby@mrao.cam.ac.uk

potentially allowing us to infer the spectrum of initial perturbations in the early universe and to determine the standard cosmological parameters to high accuracy. An obvious question to ask is whether there are any signatures of extra dimensions which could be imprinted on the cosmic microwave sky.

The aim of this paper is to set up the evolution and constraint equations for perturbations in a cold dark matter (CDM) brane cosmology, presenting them in such a way that they can be readily compared with the standard four-dimensional results, and to provide approximate solutions in the high and low-energy universe under certain restrictions on how the bulk reacts on the brane. Our equations are clearly incomplete since they lack a propagation equation for the non-local anisotropic stress that arises from projecting the bulk Weyl tensor onto the brane, and our solutions are only valid under the neglect of this stress. However, our presentation is such that we can easily include effective four-dimensional propagation equations for the non-local stress should such equations arise from a study of the full bulk perturbations. The lack of a four-dimensional propagation equation for the non-local stress means that it is currently not possible to obtain general results for the anisotropy of the CMB in braneworld models. Such a calculation would require solving the full five-dimensional perturbation equations which is non-trivial since the equations can only be reduced to two-dimensional partial differential equations on Fourier transforming the 3-space dependence. Only qualitative results are currently known, obtained with either the standard metric-based (gauge-invariant) approach [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], or with 1+3 covariant methods [21, 22, 23].

In order to make this paper self-contained, we begin by giving a brief overview of the 1+3 covariant approach to cosmology and define the key variables we use to characterize the perturbations in Sec. II. After a short review of how this approach can be used to describe the general dynamics of Randall-Sundrum braneworlds, in Sec. IV we present a complete set of frame-independent, linear equations describing the evolution of the total matter variables and the non-local energy and momentum densities for scalar perturbations in an almost-FRW universe (with arbitrary spatial curvature). Many of these equations, which employ only covariantly defined, gauge-invariant variables, have simple Newtonian analogues [24], and their physical meanings are considerably more transparent than those that underlie more standard metric-based approaches. In Sec. VII we derive analytic solutions for the scalar modes in the high and low-energy radiation-dominated universe neglecting the non-local anisotropic stress. In principle, these solutions could be used in a phenomenological manner to generate more general solutions which include non-local anisotropic stress, using Green's method and an ansatz for the non-local stress. Such a study will be presented in a subsequent paper [25], along with a numerical calculation of the CMB power spectrum employing this phenomenology. As a first step, we derive the covariant line of sight solution for the temperature anisotropies from scalar modes in braneworld models in Sec. VIII and present some comments on how higher dimensional effects may remain in the low-energy universe and thus imprint on the microwave sky. An appendix presents the scalar perturbation equations in the matter energy frame during radiation domination.

II. 1+3 COVARIANT DECOMPOSITION

Throughout this paper we adopt the metric signature $(-+++)$. Our conventions for the Riemann and Ricci tensor are fixed by $[\nabla_a, \nabla_b]U_c = R_{abc}{}^d U_d$ and $R_{ab} = R_{acb}{}^c$. Lowercase latin indices $a \dots b$ are used to denote the standard 4-dimensional (1+3) spacetime whereas uppercase $A \dots B$ and the tilde of any physical quantity are used to denote 5-dimensional (1+4) spacetime (of the braneworld). Round (square) bracket denote symmetrization (antisymmetrization) on the enclosed indices. We use units with $\hbar = c = 1$, so the 4-dimensional gravitational constant is related to the 4-dimensional Planck mass via $G = M_P{}^{-2}$.

We start by choosing a 4-velocity u^a. This must be physically defined in such a way that if the universe is exactly FRW, the velocity reduces to that of the fundamental observers to ensure gauge-invariance of the approach. From the 4-velocity u^a, we construct a projection tensor h_{ab} into the space perpendicular to u^a (the instantaneous rest space of observers whose 4-velocity is u^a):

$$h_{ab} \equiv g_{ab} + u_a u_b \,, \tag{II.1}$$

where g_{ab} is the metric of the spacetime. The operation of projecting a tensor fully with h_{ab}, symmetrizing, and removing the trace on every index (to return the projected-symmetric-trace-free or PSTF part) is denoted by angle brackets, i.e. $T_{\langle ab \dots c \rangle}$.

The symmetric tensor h_{ab} is used to define a projected (spatial) covariant derviative D^a which when acting on a tensor $T^{b \dots c}{}_{d \dots e}$ returns a tensor that is orthogonal to u^a on every index,

$$D^a T^{b \dots c}{}_{d \dots e} \equiv h^a{}_p h^b{}_r \dots h^c{}_s h^t{}_d \dots h^u{}_e \nabla^p T^{r..s}{}_{t..u} \,, \tag{II.2}$$

where ∇^a denotes the usual covariant derivative.

The covariant derivative of the 4-velocity can be decomposed as follows:

$$\nabla_a u_b = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\Theta h_{ab} - u_a A_b \ , \tag{II.3}$$

where ω_{ab} is the vorticity which satisfies $u^a\omega_{ab} = 0$, $\sigma_{ab} = \sigma_{\langle ab\rangle}$ is the shear which is PSTF, $\Theta \equiv \nabla^a u_a = 3H$ measures the volume expansion rate (where H is the local Hubble parameter), and $A_a \equiv u^b\nabla_b u_a$ is the acceleration.

Gauge-invariant quantities can be constructed from scalar variables by taking their projected gradients. Such quantities vanish in the FRW limit by construction. The comoving fractional projected gradient of the density field $\rho^{(i)}$ of a species i (for example, photons) is one important example of this construction:

$$\Delta_a^{(i)} \equiv \frac{a}{\rho^{(i)}} D_a \rho^{(i)} \ , \tag{II.4}$$

where a is a locally defined scale factor satisfying

$$\dot{a} \equiv u^b \nabla_b a = Ha \ , \tag{II.5}$$

which is included to remove the effect of the expansion on the projected gradients. Another important vector variable is the comoving projected gradient of the expansion,

$$\mathcal{Z}_a \equiv a D_a \Theta \ , \tag{II.6}$$

which provides a measure of the inhomogeneity of the expansion.

The matter stress-energy tensor T_{ab} can be decomposed irreducibly with respect to u^a as follows:

$$T_{ab} \equiv \rho u_a u_b + 2u_{(a}q_{b)} + P h_{ab} + \pi_{ab} \ , \tag{II.7}$$

where $\rho \equiv T_{ab}u^a u^b$ is the energy density measured by an observer moving with 4-velocity u^a, $q_a \equiv -h^b{}_a T_{bc} u^c$ is the energy flux or momentum density (orthogonal to u^a), $P \equiv h_{ab}T^{ab}/3$ is the isotropic pressure, and the PSTF tensor $\pi_{ab} \equiv T_{\langle ab\rangle}$ is the anisotropic stress.

The remaining gauge-invariant variables are formed from the Weyl tensor C_{abcd} which vanishes in an exact FRW universe because these models are conformally flat. The ten degrees of freedom in the 4-dimensional Weyl tensor can be encoded in two PSTF tensors: the electric and magnetic parts defined respectively as

$$E_{ab} = C_{abcd}u^b u^d \ , \tag{II.8}$$

$$H_{ab} = \frac{1}{2} C_{acst} u^c \eta^{st}{}_{bd} u^d \ , \tag{II.9}$$

where η_{abcd} is the 4-dimensional covariant permutation tensor.

III. FIELD EQUATIONS OF THE BRANEWORLD

In a recent paper, Maartens [22] introduced a formalism for describing the non-linear, intrinsic dynamics of the brane in Randall-Sundrum type II braneworld models in the form of bulk corrections to the 1+3 covariant propagation and constraint equations of general relativity. This approach is well suited to identifying the geometric and physical properties which determine homogeneity and anisotropy on the brane, and serves as a basis for developing a gauge-invariant description of cosmological perturbations in these models.

An important distinction between braneworlds and general relativity is that the set of 1+3 dynamical equations does not close on the brane. This is because there is no propagation equation for the non-local effective anistropic stress that arises from projecting the bulk Weyl tensor onto the brane. The physical implication is that the initial value problem cannot be solved by brane-bound observers. The non-local Weyl variables enter crucially into the dynamics (for example, the Raychaudhuri equation) of the intrinsic geometry of the brane. Consequently, the existence of these non-local effects leads to the violation of several important results in theoretical cosmology, such as the connection between isotropy of the CMB and the Robertson-Walker geometry.

The field equations induced on the brane are derived by Shiromizu et al [26] using the Gauss-Codazzi equations, together with the Israel junction conditions and Z_2 symmetry. The standard Einstein equation is modified with new terms carrying the bulk effects on the brane:

$$G_{ab} = -\Lambda g_{ab} + \kappa^2 T_{ab} + \tilde{\kappa}^4 \mathcal{S}_{ab} - \mathcal{E}_{ab} \ , \tag{III.1}$$

where $\kappa^2 = 8\pi/{M_p}^2$. The energy scales are related to each other via

$$\lambda = 6\frac{\kappa^2}{\tilde{\kappa}^4} \,, \tag{III.2}$$

$$\Lambda = \frac{1}{2}\tilde{\kappa}^2\left(\tilde{\Lambda} + \frac{1}{6}\tilde{\kappa}^2\lambda^2\right) \,, \tag{III.3}$$

where $\tilde{\Lambda}$ is the cosmological constant in the bulk and λ is the tension of the brane. The bulk corrections to the Einstein equations on the brane are made up of two parts: (i) the matter fields which contribute local quadratic energy-momentum corrections via the symmetric tensor $\mathcal{S}_{ab}$; and (ii) the non-local effects from the free gravitational field in the bulk transmitted by the (symmetric) projection $\mathcal{E}_{ab}$ of the bulk Weyl tensor. The matter corrections are given by

$$\mathcal{S}_{ab} = \frac{1}{12}T_c{}^cT_{ab} - \frac{1}{4}T_{ac}T^c{}_b + \frac{1}{24}g_{ab}[3T_{cd}T^{cd} - (T_c{}^c)^2] \,. \tag{III.4}$$

We note that the local part of the bulk gravitational field is the five dimensional Einstein tensor $\tilde{G}_{AB}$, which is determined by the bulk field equations. Consequently, $\mathcal{E}_{ab}$ transmits non-local gravitational degrees of freedom from the bulk to the brane that includes both tidal (or Coulomb), gravito-magnetic, and transverse traceless (gravitational wave) effects.

The bulk corrections can all be consolidated into an effective total energy density, pressure, anisotropic stress and energy flux. The modified Einstein equations take the standard Einstein form with a re-defined energy-momentum tensor:

$$G_{ab} = -\Lambda g_{ab} + \kappa^2 T^{\rm tot}_{ab} \,, \tag{III.5}$$

where

$$T^{\rm tot}_{ab} = T_{ab} + \frac{\tilde{\kappa}^4}{\kappa^2}\mathcal{S}_{ab} - \frac{1}{\kappa^2}\mathcal{E}_{ab} \,. \tag{III.6}$$

Decomposing $\mathcal{E}_{ab}$ irreducibly with respect to u^a by analogy with Eq. (II.7) [21, 22, 23],

$$\mathcal{E}_{ab} = -\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\left(\mathcal{U}u_au_b + 2u_{(a}\mathcal{Q}_{b)} + \frac{\mathcal{U}}{3}h_{ab} + \mathcal{P}_{ab}\right), \tag{III.7}$$

(the prefactor is included to make e.g. $\mathcal{U}$ have dimensions of energy density), it follows that the total density, pressure, energy flux and anisotropic pressure are given as follows:

$$\rho^{\rm tot} = \rho + \frac{\tilde{\kappa}^4}{\kappa^6}\left[\frac{\kappa^4}{24}(2\rho^2 - 3\pi^{ab}\pi_{ab}) + \mathcal{U}\right] \,, \tag{III.8}$$

$$P^{\rm tot} = P + \frac{\tilde{\kappa}^4}{\kappa^6}\left[\frac{\kappa^4}{24}\left(2\rho^2 + 4P\rho + \pi^{ab}\pi_{ab} - 4q_aq^a\right) + \frac{1}{3}\mathcal{U}\right] \,, \tag{III.9}$$

$$q^{\rm tot}_a = q_a + \frac{\tilde{\kappa}^4}{\kappa^6}\left[\frac{\kappa^4}{24}(4\rho q_a - 6\pi_{ab}q^b) + \mathcal{Q}_a\right] \,, \tag{III.10}$$

$$\pi^{\rm tot}_{ab} = \pi_{ab} + \frac{\tilde{\kappa}^4}{\kappa^6}\left[\frac{\kappa^4}{12}\left\{-(\rho + 3P)\pi_{ab} - 3\pi_{c\langle a}\pi_{b\rangle}{}^c + 3q_{\langle a}q_{b\rangle}\right\} + \mathcal{P}_{ab}\right] \,. \tag{III.11}$$

For the braneworld case, it is useful to introduce an additional dimensionless gradient which describes inhomogeneity in the non-local energy density $\mathcal{U}$:

$$\Upsilon_a \equiv \frac{a}{\rho}D_a\mathcal{U} \,. \tag{III.12}$$

The Gauss-Codazzi scalar equation for the 3-curvature defined by $\mathcal{R}$ is given by

$$\mathcal{R} = 2\kappa^2\rho + \frac{1}{6}\tilde{\kappa}^4\rho^2 + 2\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\mathcal{U} - \frac{2}{3}\Theta^2 + 2\Lambda \,, \tag{III.13}$$

where

$$\mathcal{R} \equiv {}^{(3)}R = h^{ab}\ {}^{(3)}R_{ab} \tag{III.14}$$

with ${}^{(3)}R_{ab}$ the intrinsic curvature of the surfaces orthogonal to u^a[1]. In FRW models the Gauss-Codazzi constraint reduces to the modified Friedmann equation

$$H^2 + \frac{K}{a^2} = \frac{1}{3}\kappa^2\rho + \frac{1}{3}\Lambda + \frac{1}{36}\tilde{\kappa}^4\rho^2 + \frac{1}{3}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \mathcal{U}, \tag{III.15}$$

where the 3-curvature scalar is $\mathcal{R} = 6K/a^2$. In non-flat models ($K \neq 0$) $\mathcal{R}$ is not gauge-invariant since it does not vanish in the FRW limit. However, the comoving projected gradient

$$\eta_b \equiv \frac{a}{2} D_b \mathcal{R} \tag{III.16}$$

is a gauge-invariant measure of inhomogeneity in the intrinsic three curvature of the hypersurfaces orthogonal to u^a.

IV. LINEARISED SCALAR PERTURBATION EQUATIONS FOR THE TOTAL MATTER VARIABLES

A. Local and non-local conservation equations

Based on the form of the bulk energy-momentum tensor and Z_2 symmetry, the brane energy-momentum tensor is still covariantly conserved:

$$\nabla^b T_{ab} = 0\ . \tag{IV.1}$$

The contracted Bianchi identities on the brane ensure conservation of the total energy-momentum tensor, which combined with conservation of the matter tensor gives

$$\nabla^a \mathcal{E}_{ab} = \tilde{\kappa}^4 \nabla^a \mathcal{S}_{ab}\ . \tag{IV.2}$$

The longitudinal part of $\mathcal{E}_{ab}$ is sourced by quadratic energy-momentum terms including spatial gradients and time derivatives. As a result any evolution and inhomogeneity in the matter fields would generate non-local Coulomb-like gravitational effects in the bulk which back react on the brane. The conservation equation (IV.1) implies evolution equations for the energy and momentum densities, and these are unchanged from their general relativistic form. To linear order in an almost-FRW brane cosmology we have

$$\dot{\rho} + \Theta(\rho + P) + D^a q_a = 0\ , \tag{IV.3}$$

and

$$\dot{q}_a + \frac{4}{3}\Theta q_a + (\rho + P)A_a + D_a P + D^b \pi_{ab} = 0\ . \tag{IV.4}$$

The linearised propagation equations for $\mathcal{U}$ and $\mathcal{Q}$ follow from Eq. (IV.2) (see Ref. [22]):

$$\dot{\mathcal{U}} + \frac{4}{3}\Theta\mathcal{U} + D^a \mathcal{Q}_a = 0\ , \tag{IV.5}$$

and

$$\dot{\mathcal{Q}}_a + \frac{4}{3}\Theta\mathcal{Q}_a + \frac{1}{3}D_a\mathcal{U} + D^b\mathcal{P}_{ab} + \frac{4}{3}\mathcal{U}A_a = \frac{\kappa^4}{12}(\rho + P)\left(-2D_a\rho + 3D^b\pi_{ab} + 2\Theta q_a\right)\ . \tag{IV.6}$$

Taking the projected derivative of Eq. (IV.3) we obtain the propagation equation for Δ_a at linear order:

$$\rho\dot{\Delta}_a + (\rho + P)(\mathcal{Z}_a + a\Theta A_a) + aD_a D^b q_b + a\Theta D_a P - \Theta P \Delta_a = 0. \tag{IV.7}$$

---

[1] If the vorticity is non-vanishing flow-orthogonal hypersurfaces will not exist, and $\mathcal{R}$ cannot be interpreted as the spatial curvature scalar.

From equation (IV.5), we obtain the evolution equation of the spatial gradient of the non-local energy density:

$$\dot{\Upsilon}_a = \left(\frac{P}{\rho} - \frac{1}{3}\right)\Theta\Upsilon_a - \frac{4}{3}\frac{\mathcal{U}}{\rho}(\mathcal{Z}_a + a\Theta A_a) - \frac{a}{\rho}D_a D^b \mathcal{Q}_b. \tag{IV.8}$$

From the propagation equations for $\mathcal{U}$ and $\mathcal{Q}$ it can be seen that the energy of the projected Weyl fluid is conserved while the momentum is not conserved; rather it is driven by the matter source terms on the right of Eq. (IV.2). Note that no propagation equation for $\mathcal{P}_{ab}$ is implied so the set of equations will not close.

B. Propagation and constraint equations

In this section we give the linearised gravito-magnetic and gravito-electric propagation and constraint scalar equations on the brane, which follow from the Bianchi identities, and the equations for the kinematic variables σ_{ab}, and Θ and its gradient $\mathcal{Z}_a$ which follow from the Ricci identity.

For scalar perturbations, the magnetic part of the Weyl tensor H_{ab} and the vorticity tensor ω_{ab} vanish identically. The electric part of the Weyl tensor E_{ab} and the shear σ_{ab} need not vanish. The non-vanishing variables satisfy the following propagation and constraint equations on the brane:

1. Gravito-electric propagation:

$$\begin{aligned}&\dot{E}_{ab} + \Theta E_{ab} + \frac{1}{2}\kappa^2(\rho+P)\sigma_{ab} + \frac{1}{2}\kappa^2 D_{\langle a}q_{b\rangle} + \frac{1}{6}\kappa^2\Theta\pi_{ab} + \frac{1}{2}\kappa^2\dot{\pi}_{ab}\\ &= \frac{1}{72}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 [\kappa^4\{-6\rho(\rho+P)\sigma_{ab} + 3(\dot{\rho}+3\dot{P})\pi_{ab} + 3(\rho+3P)\dot{\pi}_{ab}\\ &- 6\rho D_{\langle a}q_{b\rangle} + \Theta[\rho+3P]\pi_{ab}\} - 48\mathcal{U}\sigma_{ab} - 36\dot{\mathcal{P}}_{ab} - 36D_{\langle a}\mathcal{Q}_{b\rangle} - 12\Theta\mathcal{P}_{ab}]\ ;\end{aligned} \tag{IV.9}$$

2. Shear propagation:

$$\dot{\sigma}_{ab} + \frac{2}{3}\Theta\sigma_{ab} + E_{ab} - \frac{1}{2}\kappa^2\pi_{ab} - D_{\langle a}A_{b\rangle} = \frac{1}{24}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\{\kappa^4[-(\rho+3P)\pi_{ab}] + 12\mathcal{P}_{ab}\}\ ; \tag{IV.10}$$

3. Shear constraint:

$$D^b\sigma_{ab} - \frac{2}{3}D_a\Theta + \kappa^2 q_a = -\frac{1}{6}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4(\kappa^4\rho q_a + 6\mathcal{Q}_a)\ ; \tag{IV.11}$$

4. Gravito-electric divergence:

$$\begin{aligned}D^bE_{ab} + \frac{1}{2}\kappa^2 D^b\pi_{ab} - \frac{1}{3}\kappa^2 D_a\rho + \frac{1}{3}\kappa^2\Theta q_a = \frac{1}{48}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\Bigg[&\kappa^4\left(-\frac{8}{3}\rho\Theta q_a + 2(\rho+3P)D^b\pi_{ab} + \frac{8}{3}\rho D_a\rho\right)\\ &+ 16D_a\mathcal{U} - 16\Theta\mathcal{Q}_a - 24D^b\mathcal{P}_{ab}\Bigg]\ ;\end{aligned} \tag{IV.12}$$

5. Modified Raychaudhuri equation:

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}\kappa^2(\rho+3P) + \Lambda - \frac{1}{12}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\kappa^4\rho(2\rho+3P) + 12\mathcal{U}] + D^aA_a\ ; \tag{IV.13}$$

6. Propagation equation for the comoving expansion gradient $\mathcal{Z}_a$ which follows from Eq. (IV.13):

$$\dot{\mathcal{Z}}_a + \frac{2}{3}\Theta\mathcal{Z}_a - a\dot{\Theta}A_a + \frac{\kappa^2}{2}aD_a(\rho+3P) - aD_aD^bA_b = -\frac{1}{12}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\{\kappa^4 aD_a[\rho(2\rho+3P)] + 12aD_a\mathcal{U}\}\ . \tag{IV.14}$$

The spatial gradient of the 3-curvature scalar is an auxiliary variable. It can be related to the other gauge-invariant variables using Eqs. (III.13) and (III.16):

$$\eta_a = \kappa^2 \rho \Delta_a + \frac{1}{6}\tilde{\kappa}^4 \rho^2 \Delta_a + \left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \rho \Upsilon_a - \frac{2}{3}\Theta \mathcal{Z}_a \,. \tag{IV.15}$$

Taking the time derivative of Eq. (IV.15), commuting the spatial and temporal derivatives, and then making use of Eqs. (IV.13) and (IV.14), we obtain the evolution of the spatial gradient of the 3-curvature scalar:

$$\dot{\eta}_a + \frac{2}{3}\Theta \eta_a + \frac{1}{3}\mathcal{R}(\mathcal{Z}_a + a\Theta A_a) + \frac{2}{3}\Theta a D_a D^b A_b = -\left(\kappa^2 + \frac{1}{6}\tilde{\kappa}^4 \rho\right) a D_a D^b q_b - \left(\frac{\tilde{\kappa}}{\kappa}\right)^4 a D_a D^b \mathcal{Q}_b. \tag{IV.16}$$

In general relativity, propagating η_a is a useful device to avoid numerical instability problems when solving for isocurvature modes in a zero acceleration frame (such as the rest-frame of the CDM) [27].

V. COSMOLOGICAL SCALAR PERTURBATIONS IN THE BRANEWORLD

The tensor-valued, partial differential equations presented in the earlier sections can be reduced to scalar-valued, ordinary differential equations by expanding in an appropriate complete set of eigentensors. For scalar perturbations all gauge-invariant tensors can be constructed from derivatives of scalar functions. It is thus natural to expand in STF tensors derived from the scalar eigenfunctions $Q^{(k)}$ of the projected Laplacian:

$$D^2 Q^{(k)} = -\frac{k^2}{a^2} Q^{(k)} \,, \tag{V.1}$$

satisfying $\dot{Q}^{(k)} = O(1)^2$. We adopt the following harmonic expansions of the gauge-invariant variables:

$$\begin{aligned}
\Delta_a^{(i)} &= \sum_k k \Delta_k^{(i)} Q_a^{(k)} \,, & \mathcal{Z}_a &= \sum_k \frac{k^2}{a} \mathcal{Z}_k Q_a^{(k)} \,, \\
q_a^{(i)} &= \rho^{(i)} \sum_k q_k^{(i)} Q_a^{(k)} \,, & \pi_{ab}^{(i)} &= \rho^{(i)} \sum_k \pi_k^{(i)} Q_{ab}^{(k)} \,, \\
E_{ab} &= \sum_k \frac{k^2}{a^2} \Phi_k Q_{ab}^{(k)} \,, & \sigma_{ab} &= \sum_k \frac{k}{a} \sigma_k Q_{ab}^{(k)} \,, \\
v_a^{(i)} &= \sum_k v_k^{(i)} Q_a^{(k)} \,, & A_a &= \sum_k \frac{k}{a} A_k Q_a^{(k)} \,.
\end{aligned} \tag{V.2}$$

Here $v_a^{(i)}$ is the 3-velocity of species i relative to u^a; for the CDM model considered here we shall make use of $v_a^{(i)}$ for baryons b and CDM c. For photons γ and neutrinos ν we continue to work with the momentum densities which are related to the peculiar velocity of the energy frame for that species by e.g. $q_a^{(\gamma)} = (4/3)\rho^{(\gamma)} v_a^{(\gamma)}$ in linear theory. The scalar expansion coefficients, such as $\Delta_k^{(i)}$ are first-order gauge-invariant variables satisfying e.g. $D^a \Delta_k^{(i)} = O(2)$. Projected vectors $Q_a^{(k)}$ and STF tensors $Q_{ab}^{(k)}$ are defined by [28]:

$$\begin{aligned}
Q_a^{(k)} &= -\frac{a}{k} D_a Q^{(k)} \,, \\
Q_{ab}^{(k)} &= \frac{a^2}{k^2} D_{\langle a} D_{b\rangle} Q^{(k)} \,,
\end{aligned} \tag{V.3}$$

so that

$$D^b Q_{ab}^{(k)} = \frac{2}{3}\left(\frac{k}{a}\right)\left(1 - \frac{3K}{k^2}\right) Q_a^{(k)} \,. \tag{V.4}$$

---

[2] The notation $O(n)$ is short for $O(\epsilon^n)$ where ϵ is some dimensionless quantity characterising the departure from FRW symmetry.

We expand the non-local perturbation variables in scalar harmonics in the following manner:

$$\begin{aligned}\Upsilon_a &= \sum_k k\Upsilon_k Q_a^{(k)} \ , \\ \mathcal{Q}_a &= \sum_k \rho\mathcal{Q}_k Q_a^{(k)} \ , \\ \mathcal{P}_{ab} &= \sum_k \rho\mathcal{P}_k Q_{ab}^{(k)} \ .\end{aligned} \tag{V.5}$$

In addition, we can expand the projected gradient of the 3-curvature term:

$$\eta_a = \sum_k 2\left(\frac{k^3}{a^2}\right)\left(1-\frac{3K}{k^2}\right)\eta_k Q_a^{(k)} \ . \tag{V.6}$$

The form of this expansion is chosen so that if we adopt the energy frame (where $q_a = 0$) the variable η_k coincides with the curvature perturbation usually employed in gauge-invariant calculations.

A. Scalar equations on the brane

It is now straightforward to expand the 1+3 covariant propagation and constraint equations in scalar harmonics. We shall consider the CDM model where the particle species are baryons (including electrons), which we model as an ideal fluid with pressure $p^{(b)}$ and peculiar velocity $v_a^{(b)}$, cold dark matter, which has vanishing pressure and peculiar velocity $v_a^{(c)}$, and photons and (massless) neutrinos which require a kinetic theory description. We neglect photon polarization, although this can easily be included in the 1+3 covariant framework [29]. Also, we assume that the entropy perturbations are negligible for the baryons, so that $D_a P^{(b)} = c_s^2 D_a \rho^{(b)}$ where c_s^2 is the adiabatic sound speed. A complete set of 1+3 perturbation equations for the general relativistic model can be found in [30]. We extend these equations to braneworld models here.

In the following, perturbations in the total matter variables are related to those in the individual components by

$$\rho\Delta_k = \sum_i \rho^{(i)}\Delta_k^{(i)}, \quad \rho q_k = \sum_i \rho^{(i)} q_k^{(i)}, \quad \rho\pi_k = \sum_k \rho^{(i)}\pi_k^{(i)}, \tag{V.7}$$

where $q_k^{(b)} = (1+P^{(b)}/\rho^{(b)})v_k^{(b)}$, $q_k^{(c)} = v_k^{(c)}$, and $\pi_k^{(i)}$ vanishes for baryons and CDM. Similarly, the total density and pressure are obtained by summing over components, e.g. $P = \sum_i P^{(i)}$. It is also convenient to write $P = (\gamma-1)\rho$, but γ should not be assumed constant (in space or time).

We begin with the equation for the gravito-electric field:

$$\begin{aligned}&\left(\frac{k}{a}\right)^2\left(\dot\Phi_k + \frac{1}{3}\Theta\Phi_k\right) + \frac{1}{2}\frac{k}{a}\kappa^2\rho(\gamma\sigma_k - q_k) + \frac{1}{6}\kappa^2\rho\Theta(1-3\gamma)\pi_k + \frac{1}{2}\kappa^2\rho\dot\pi_k \\ &= \frac{1}{72}\left(\frac{\tilde\kappa}{\kappa}\right)^4\Bigg\{-\kappa^4\left[6\left(\frac{k}{a}\right)\rho^2(\gamma\sigma_k - q_k) - 3(\dot\rho + 3\dot P)\rho\pi_k - 3(3\gamma-2)\rho(\rho\dot\pi_k + \dot\rho\pi_k) - (3\gamma-2)\rho^2\Theta\pi_k\right] \\ &- 12\left(\frac{k}{a}\right)(4\mathcal{U}\sigma_k - 3\rho\mathcal{Q}_k) - 36(\dot\rho\mathcal{P}_k + \rho\dot{\mathcal{P}}_k) - 12\rho\Theta\mathcal{P}_k\Bigg\} \ .\end{aligned} \tag{V.8}$$

We have written this equation in such a form that every term is manifestly frame-independent. The shear propagation equation is

$$\frac{k}{a}\left(\dot\sigma_k + \frac{1}{3}\Theta\sigma_k\right) + \left(\frac{k}{a}\right)^2(\Phi_k + A_k) - \frac{\kappa^2}{2}\rho\pi_k = \frac{1}{24}\left(\frac{\tilde\kappa}{\kappa}\right)^4\left[-(3\gamma-2)\kappa^4\rho^2\pi_k + 12\rho\mathcal{P}_k\right] . \tag{V.9}$$

The shear constraint is given by

$$\kappa^2\rho q_k - \frac{2}{3}\left(\frac{k}{a}\right)^2\left[\mathcal{Z}_k - \left(1-\frac{3K}{k^2}\right)\sigma_k\right] = -\frac{1}{6}\left(\frac{\tilde\kappa}{\kappa}\right)^4(\kappa^4\rho^2 q_k + 6\rho\mathcal{Q}_k). \tag{V.10}$$

The gravito-electric divergence is

$$
\begin{aligned}
&2\left(\frac{k}{a}\right)^3\left(1-\frac{3K}{k^2}\right)\Phi_k-\kappa^2\rho\left(\frac{k}{a}\right)\left[\Delta_k-\left(1-\frac{3K}{k^2}\right)\pi_k\right]+\kappa^2\Theta\rho q_k \\
&=\frac{1}{16}\left(\frac{\tilde\kappa}{\kappa}\right)^4\left\{\kappa^4\left[-\frac{8}{3}\rho^2\Theta q_k+\frac{4}{3}(3\gamma-2)\rho^2\left(1-\frac{3K}{k^2}\right)\frac{k}{a}\pi_k+\frac{8}{3}\frac{k}{a}\rho^2\Delta_k\right]\right. \\
&\left.+16\frac{k}{a}\rho\Upsilon_k-16\Theta\rho\mathcal{Q}_k-16\rho\left(\frac{k}{a}\right)\left(1-\frac{3K}{k^2}\right)\mathcal{P}_k\right\}.
\end{aligned}
\tag{V.11}
$$

The propagation equation for the comoving expansion gradient $\mathcal{Z}_a$ is given by

$$
\begin{aligned}
&\dot{\mathcal{Z}}_k+\frac{1}{3}\Theta\mathcal{Z}_k-\frac{a}{k}\dot\Theta A_k+\frac{k}{a}A_k+\frac{\kappa^2}{2}\frac{a}{k}\left[2(\rho^{(\gamma)}\Delta_k^{(\gamma)}+\rho^{(\nu)}\Delta_k^{(\nu)})+(1+3c_s^2)\rho^{(b)}\Delta_k^{(b)}+\rho^{(c)}\Delta_k^{(c)}\right] \\
&=-\frac{1}{12}\left(\frac{\tilde\kappa}{\kappa}\right)^4\frac{a}{k}\left\{\kappa^4[(2\rho+3P)\rho\Delta_k+\rho(3\rho^{(\gamma)}\Delta_k^{(\gamma)}+3\rho^{(\nu)}\Delta_k^{(\nu)}+(2+3c_s^2)\rho^{(b)}\Delta_k^{(b)}+2\rho^{(c)}\Delta_k^{(c)})]+12\rho\Upsilon_k\right\}.
\end{aligned}
\tag{V.12}
$$

The non-local evolution equations for Υ_k and $\mathcal{Q}_k$ are

$$
\dot\Upsilon_k=\frac{1}{3}(3\gamma-4)\Theta\Upsilon_k-\frac{4}{3}\Theta\frac{\mathcal{U}}{\rho}A_k-\frac{4}{3}\frac{\mathcal{U}}{\rho}\frac{k}{a}\mathcal{Z}_k+\frac{k}{a}\mathcal{Q}_k,
\tag{V.13}
$$

and

$$
\dot{\mathcal{Q}}_k-\frac{1}{3}(3\gamma-4)\Theta\mathcal{Q}_k+\frac{1}{3}\frac{k}{a}\left[\Upsilon_k+2\left(1-\frac{3K}{k^2}\right)\mathcal{P}_k\right]+\frac{4}{3}\frac{k}{a}\frac{\mathcal{U}}{\rho}A_k=\frac{\kappa^4}{6}\gamma\rho\left\{\Theta q_k+\frac{k}{a}\left[\left(1-\frac{3K}{k^2}\right)\pi_k-\Delta_k\right]\right\}.
\tag{V.14}
$$

The spatial gradient of the 3-curvature scalar is

$$
\left(\frac{k}{a}\right)^2\left(1-\frac{3K}{k^2}\right)\eta_k=\frac{\kappa^2\rho}{2}\Delta_k+\frac{\tilde\kappa^4\rho^2}{12}\Delta_k+\frac{1}{2}\left(\frac{\tilde\kappa}{\kappa}\right)^4\rho\Upsilon_k-\frac{1}{3}\frac{k}{a}\Theta\mathcal{Z}_k,
\tag{V.15}
$$

and it evolves according to

$$
\frac{k}{a}\left(1-\frac{3K}{k^2}\right)\left(\dot\eta_k-\frac{1}{3}\Theta A_k\right)+\frac{K}{a^2}\mathcal{Z}_k-\frac{1}{2}\kappa^2\rho q_k=\frac{1}{12}\left(\frac{\tilde\kappa}{\kappa}\right)^4(\kappa^4\rho^2q_k+6\rho\mathcal{Q}_k).
\tag{V.16}
$$

The evolution equations for the scalar harmonic components of the comoving, fractional density gradients for photons, neutrinos, baryons and cold dark matter (CDM) are

$$
\dot\Delta_k^{(\gamma)}=-\frac{k}{a}\left(\frac{4}{3}\mathcal{Z}_k-q_k^{(\gamma)}\right)-\frac{4}{3}\Theta A_k\quad\text{(photons)},
\tag{V.17}
$$

$$
\dot\Delta_k^{(\nu)}=-\frac{k}{a}\left(\frac{4}{3}\mathcal{Z}_k-q_k^{(\nu)}\right)-\frac{4}{3}\Theta A_k\quad\text{(neutrinos)},
\tag{V.18}
$$

$$
\dot\Delta_k^{(b)}=\left(1+\frac{P^{(b)}}{\rho^{(b)}}\right)\left[-\frac{k}{a}(\mathcal{Z}_k-v_k^{(b)})-\Theta A_k\right]+\left(\frac{P^{(b)}}{\rho^{(b)}}-c_s^2\right)\Theta\Delta_k^{(b)}\quad\text{(baryons)},
\tag{V.19}
$$

$$
\dot\Delta_k^{(c)}=-\frac{k}{a}(\mathcal{Z}_k-v_k^{(c)})-\Theta A_k\quad\text{(CDM)}.
\tag{V.20}
$$

The evolution equations for the momentum densities and peculiar velocities are

$$
\dot q_k^{(\gamma)}=-\frac{1}{3}\frac{k}{a}\left[\Delta_k^{(\gamma)}+4A_k+2\left(1-\frac{3K}{k^2}\right)\pi_k^{(\gamma)}\right]+n_e\sigma_T\left(\frac{4}{3}v_k^{(b)}-q_k^{(\gamma)}\right),
\tag{V.21}
$$

$$
\dot q_k^{(\nu)}=-\frac{1}{3}\frac{k}{a}\left[\Delta_k^{(\nu)}+4A_k+2\left(1-\frac{3K}{k^2}\right)\pi_k^{(\nu)}\right],
\tag{V.22}
$$

$$
(\rho^{(b)}+P^{(b)})\dot v_k^{(b)}=-(\rho^{(b)}+P^{(b)})\left[\frac{1}{3}(1-3c_s^2)\Theta v_k^{(b)}+\frac{k}{a}A_k\right]-\frac{k}{a}(1+c_s^2)\Delta_k^{(b)}-n_e\sigma_T\frac{\rho^{(\gamma)}}{\rho^{(\nu)}}\left(\frac{4}{3}v_k^{(b)}-q_k^{(\gamma)}\right),
\tag{V.23}
$$

$$
\dot v_k^{(c)}=-\frac{1}{3}\Theta v_k^{(c)}-\frac{k}{a}A_k,
\tag{V.24}
$$

where the Thomson scattering terms involving the electron density n_e and Thomson cross section σ_T arise from the interaction between photons and the tightly-coupled baryon/electron fluid. The remaining equations are the propagation equations for the anisotropic stresses of photons and neutrinos, and the higher moments of their distribution functions. These equations can be found in [30], and with polarization included in [29], since they are unchanged from general relativity. However, we shall not require these additional equations at the level of approximation we make in our subsequent calculations.

VI. PERTURBATION DYNAMICS IN THE CDM FRAME

In this section we specialize our equations to FRW backgrounds that are spatially flat[3] and we ignore the effects of the cosmological constant in the early radiation-dominated universe. To solve the equations it is essential to make a choice of frame u^a. In Ref. [30] two of the present authors adopted a frame comoving with the CDM. Since the CDM is pressure free, this u^a is geodesic ($A_a = 0$) which simplifies the equations considerably. We shall adopt this frame choice here also, though we note it may be preferable to use a frame more closely tied to the dominant matter component over the epoch of interest. This can be easily accomplished by adopting the energy frame ($q_a = 0$). For completeness, we give equations in the energy frame in the appendix.

We neglect baryon pressure ($c_s^2 \to 0$ and $P^{(b)} \to 0$) and work to lowest order in the tight-coupling approximation ($n_e\sigma_T \to \infty$; see e.g. Ref. [31]). At this order the energy frame of the photons coincides with the rest frame of the baryons, so that $v_a^{(b)} = 3q_a^{(\gamma)}/(4\rho^{(\gamma)})$, and all moments of the photon distribution are vanishingly small beyond the dipole.

With these approximations and frame choice we obtain the following equations for the density perturbations of each component:

$$\dot{\Delta}_k^{(\gamma)} = -\frac{k}{a}\left(\frac{4}{3}\mathcal{Z}_k - q_k^{(\gamma)}\right) \quad \text{(photons)} , \tag{VI.1}$$

$$\dot{\Delta}_k^{(\nu)} = -\frac{k}{a}\left(\frac{4}{3}\mathcal{Z}_k - q_k^{(\nu)}\right) \quad \text{(neutrinos)} , \tag{VI.2}$$

$$\dot{\Delta}_k^{(b)} = -\frac{k}{a}(\mathcal{Z}_k - v_k^{(b)}) \quad \text{(baryons)} , \tag{VI.3}$$

$$\dot{\Delta}_k^{(c)} = -\frac{k}{a}\mathcal{Z}_k \quad \text{(CDM)} . \tag{VI.4}$$

The equations for the peculiar velocities and momentum densities are

$$(4\rho^{(\gamma)} + 3\rho^{(b)})\dot{q}_k^{(\gamma)} = -\frac{4}{3}\frac{k}{a}\rho^{(\gamma)}\Delta_k^{(\gamma)} - \rho^{(b)}\Theta q_k^{(\gamma)} , \tag{VI.5}$$

$$\dot{q}_k^{(\nu)} = -\frac{1}{3}\frac{k}{a}\left(\Delta_k^{(\nu)} + 2\pi_k^{(\nu)}\right) , \tag{VI.6}$$

along with $v_k^{(c)} = 0$ and $v_k^{(b)} = 3q_k^{(\gamma)}/4$. The latter equation, together with Eqs. (VI.1) and (VI.3), implies that $\dot{\Delta}_k^{(b)} = 3\dot{\Delta}_k^{(\gamma)}/4$ so that any entropy perturbation between the photons and baryons is conserved while tight coupling holds. The effects of baryon inertia appear in Eq. (VI.5) because of the tight coupling between the baryons and photons.

The constraint equations are found to be:

$$\kappa^2\rho q_k - \frac{2}{3}\left(\frac{k}{a}\right)^2(\mathcal{Z}_k - \sigma_k) - \frac{1}{6}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4(\kappa^4\rho^2 q_k + 6\rho\mathcal{Q}_k) = 0 , \tag{VI.7}$$

[3] More generally, curvature effects can be ignored for modes with wavelength much shorter than the curvature scale, $k \gg \sqrt{|K|}$, provided the curvature does not dominate the background dynamics.

and

$$
\begin{aligned}
&2\left(\frac{k}{a}\right)^3 \Phi_k - \kappa^2\rho\left(\frac{k}{a}\right)(\Delta_k - \pi_k) + \kappa^2\Theta\rho q_k \\
&= \frac{1}{16}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \left[\kappa^4\left(-\frac{8}{3}\Theta\rho^2 q_k + \frac{4}{3}\frac{k}{a}\rho^2\left[(3\gamma-2)\pi_k + 2\Delta_k\right]\right)\right. \\
&\left. + 16\left(\frac{k}{a}\right)\rho(\Upsilon_k - \mathcal{P}_k) - 16\Theta\rho\mathcal{Q}_k\right] .
\end{aligned}
\tag{VI.8}
$$

The propagation equation for the comoving expansion gradient in the CDM frame is

$$
\begin{aligned}
&\dot{\mathcal{Z}}_k + \frac{1}{3}\Theta\mathcal{Z}_k + \frac{\kappa^2}{2}\frac{a}{k}\left[2(\rho^{(\gamma)}\Delta_k^{(\gamma)} + \rho^{(\nu)}\Delta_k^{(\nu)}) + \rho^{(b)}\Delta_k^{(b)} + \rho^{(c)}\Delta_k^{(c)}\right] \\
&= -\frac{1}{12}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \frac{a}{k}\left\{\kappa^4[(2\rho+3P)\rho\Delta_k + \rho(3\rho^{(\gamma)}\Delta_k^{(\gamma)} + 3\rho^{(\nu)}\Delta_k^{(\nu)} + (2+3c_s^2)\rho^{(b)}\Delta_k^{(b)} + 2\rho^{(c)}\Delta_k^{(c)})] + 12\rho\Upsilon_k\right\} .
\end{aligned}
\tag{VI.9}
$$

The variables Φ_k and σ_k can be determined from the constraint equations so their propagation equations are not independent of the above set. The propagation equation for Φ_k is unchanged from Eq. (V.8) since that equation was already written in frame-invariant form. The propagation equation for the shear in the CDM frame is

$$
\frac{k}{a}\left(\dot{\sigma}_k + \frac{1}{3}\Theta\sigma_k\right) + \left(\frac{k}{a}\right)^2\Phi_k - \frac{\kappa^2}{2}\rho\pi_k = \frac{1}{24}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\left[-(3\gamma-2)\kappa^4\rho^2\pi_k + 12\rho\mathcal{P}_k\right] . \tag{VI.10}
$$

Finally we have the non-local evolution equations for Υ_k and $\mathcal{Q}_k$ which in the CDM frame become

$$
\dot{\Upsilon}_k = \frac{1}{3}(3\gamma-4)\Theta\Upsilon_k - \frac{4}{3}\frac{\mathcal{U}}{\rho}\frac{k}{a}\mathcal{Z}_k + \frac{k}{a}\mathcal{Q}_k , \tag{VI.11}
$$

and

$$
\dot{\mathcal{Q}}_k - \frac{1}{3}(3\gamma-4)\Theta\mathcal{Q}_k + \frac{1}{3}\frac{k}{a}(\Upsilon_k + 2\mathcal{P}_k) = \frac{\kappa^4}{6}\gamma\rho\left[\Theta q_k + \frac{k}{a}(\pi_k - \Delta_k)\right] . \tag{VI.12}
$$

VII. SOLUTIONS IN THE RADIATION-DOMINATED ERA

We now use the above equations to extract the mode solutions of the scalar perturbation equations in the radiation-dominated era, $\gamma = 4/3$. To simplify matters, as well as neglecting the contribution of the baryons and CDM to the background dynamics, we shall only consider those modes for which $D_a\rho^{(b)}$ and $D_a\rho^{(c)}$ make a negligible contribution to the total matter perturbation $D_a\rho$. This approximation allows us to write the total matter perturbations in the form

$$
(\rho^{(\gamma)} + \rho^{(\nu)})\Delta_k = \rho^{(\gamma)}\Delta_k^{(\gamma)} + \rho^{(\nu)}\Delta_k^{(\nu)}, \quad (\rho^{(\gamma)} + \rho^{(\nu)})q_k = \rho^{(\gamma)}q_k^{(\gamma)} + \rho^{(\nu)}q_k^{(\nu)}, \tag{VII.1}
$$

and effectively removes the back-reaction of the baryon and CDM perturbations on the perturbations of the spacetime geometry. We note that in making this approximation we lose two modes corresponding to the baryon and CDM isocurvature (density) modes of general relativity, in which the sub-dominant matter components make significant contributions to the total fractional density perturbation (which vanishes as $t \to 0$). However, for our purposes the loss of generality is not that important, while the simplifications resulting from decoupling the baryon and photon perturbations are considerable. We also neglect moments of the neutrino distribution function above the dipole (so there is no matter anisotropic stress). This approximation is good for super-Hubble modes, but fails due to neutrino free streaming on sub-Hubble scales.

We shall also assume that the non-local energy density $\mathcal{U}$ vanishes in the background for all energy regimes [21]. Physically, vanishing $\mathcal{U}$ corresponds to the background bulk being conformally flat and strictly Anti-de Sitter. Note that $\mathcal{U} = 0$ in the background need not imply that the fluctuations in the non-local energy density are zero, i.e. $\Upsilon_a \neq 0$.

With the above conditions the following set of equations are obtained:

$$\left(\frac{k}{a}\right)^2(\dot{\Phi}_k+H\Phi_k)+\frac{\kappa^2\rho}{2}\left(\frac{k}{a}\right)\left(\frac{4}{3}\sigma_k-q_k\right)\left(1+\frac{\rho}{\lambda}\right)=\frac{3}{\kappa^2}\frac{\rho}{\lambda}\left[\left(\frac{k}{a}\right)\mathcal{Q}_k+3H\mathcal{P}_k-\dot{\mathcal{P}}_k\right], \tag{VII.2}$$

$$\left(\frac{k}{a}\right)(\dot{\mathcal{Z}}_k+H\mathcal{Z}_k)+\kappa^2\rho\left(1+\frac{3\rho}{\lambda}\right)\Delta_k=-\frac{6}{\kappa^2}\frac{\rho}{\lambda}\Upsilon_k \tag{VII.3}$$

$$\left(\frac{k}{a}\right)(\dot{\sigma}_k+H\sigma_k)+\left(\frac{k}{a}\right)^2\Phi_k=\frac{3}{\kappa^2}\frac{\rho}{\lambda}\mathcal{P}_k, \tag{VII.4}$$

$$\dot{q}_k^{(\gamma)}+\frac{1}{3}\frac{k}{a}\Delta_k^{(\gamma)}=0, \tag{VII.5}$$

$$\dot{q}_k^{(\nu)}+\frac{1}{3}\frac{k}{a}\Delta_k^{(\nu)}=0, \tag{VII.6}$$

$$\dot{\Delta}_k^{(\gamma)}+\frac{k}{a}\left(\frac{4}{3}\mathcal{Z}_k-q_k^{(\gamma)}\right)=0, \tag{VII.7}$$

$$\dot{\Delta}_k^{(\nu)}+\frac{k}{a}\left(\frac{4}{3}\mathcal{Z}_k-q_k^{(\nu)}\right)=0, \tag{VII.8}$$

where recall $H=\Theta/3$. For the constraint equations we find

$$3\kappa^2\left(1+\frac{\rho}{\lambda}\right)\rho q_k-2\left(\frac{k}{a}\right)^2(\mathcal{Z}_k-\sigma_k)=-\frac{18}{\kappa^2}\frac{\rho}{\lambda}\mathcal{Q}_k, \tag{VII.9}$$

$$2\left(\frac{k}{a}\right)^3\Phi_k+\kappa^2\rho\left(1+\frac{\rho}{\lambda}\right)\left[3Hq_k-\left(\frac{k}{a}\right)\Delta_k\right]=\frac{6}{\kappa^2}\frac{\rho}{\lambda}\left[\left(\frac{k}{a}\right)(\Upsilon_k-\mathcal{P}_k)-3H\mathcal{Q}_k\right]. \tag{VII.10}$$

Finally the non-local evolution equations are found to be :

$$\dot{\Upsilon}_k=\frac{k}{a}\mathcal{Q}_k, \tag{VII.11}$$

$$9\dot{\mathcal{Q}}_k+3\left(\frac{k}{a}\right)(\Upsilon_k+2\mathcal{P}_k)=-2\kappa^4\rho\left(\frac{k}{a}\Delta_k-3Hq_k\right). \tag{VII.12}$$

It is easy to show by propagating the constraint equations that the above set of equations are consistent.

By inspection, there is a solution of these equations with

$$\Phi_k=0, \tag{VII.13}$$

$$\mathcal{Z}_k=\left[3\dot{H}\left(\frac{a}{k}\right)^2-1\right]\frac{A}{a}, \tag{VII.14}$$

$$\sigma_k=-\frac{A}{a}, \tag{VII.15}$$

$$q_k^{(\gamma)}=-\frac{4}{3}\frac{A}{a}, \tag{VII.16}$$

$$q_k^{(\nu)}=-\frac{4}{3}\frac{A}{a}, \tag{VII.17}$$

$$\Delta_k^{(\gamma)}=-4H\frac{A}{k}, \tag{VII.18}$$

$$\Delta_k^{(\nu)}=-4H\frac{A}{k}, \tag{VII.19}$$

$$\Upsilon_k=0, \tag{VII.20}$$

$$\mathcal{Q}_k=0, \tag{VII.21}$$

$$\mathcal{P}_k=0, \tag{VII.22}$$

where A is a constant. This solution describes a radiation-dominated universe that is exactly FRW except that the CDM has a peculiar velocity $\bar{v}_a^{(c)}=(A/a)Q_a^{(k)}$ relative to the velocity of the FRW fundamental observers. [This form for $\bar{v}_a^{(c)}$ clearly satisfies Eq. (V.24) with $A_a=0$.] Such a solution is possible since we have neglected the gravitational

effect of the CDM (and baryon) perturbations in making the approximations in Eq. (VII.1). The same solution arises in general relativity [30]. Including the back-reaction of the CDM perturbations, we would find additional small peculiar velocities in the dominant matter components which compensate the CDM flux. We shall not consider this irregular CDM isocurvature velocity mode any further here.

Another pair of solutions are easily found by decoupling the photon/neutrino entropy perturbations. Introducing the photon/neutrino entropy perturbation (up to constant) Δ_2 and relative flux q_2:

$$\begin{aligned} \Delta_2 &= \Delta_k^{(\gamma)} - \Delta_k^{(\nu)} \ , \\ q_2 &= q_k^{(\gamma)} - q_k^{(\nu)} \ , \end{aligned} \tag{VII.23}$$

the equations for Δ_2 and q_2 decouple to give

$$\dot{\Delta}_2 - \frac{k}{a} q_2 = 0 \ , \tag{VII.24}$$

$$\dot{q}_2 + \frac{1}{3}\frac{k}{a}\Delta_2 = 0 \ . \tag{VII.25}$$

Switching to conformal time ($d\tau = dt/a$) we can solve for Δ_2 and q_2 to find

$$q_2(\tau) = B\cos\left(\frac{k\tau}{3}\right) + C\sin\left(\frac{k\tau}{3}\right) \ , \tag{VII.26}$$

$$\Delta_2(\tau) = B\sin\left(\frac{k\tau}{3}\right) - C\cos\left(\frac{k\tau}{3}\right) \ . \tag{VII.27}$$

The constants B and C label the neutrino velocity and density isocurvature modes respectively [30, 32], in which the neutrinos and photons initially have mutually compensating peculiar velocities and density perturbations. The perfect decoupling of these isocurvature modes is a consequence of our neglecting anisotropic stresses (and higher moments of the distribution functions) and baryon inertia.

Having decoupled the entropy perturbations, we write the remaining equations in terms of the total variables Δ_k and q_k. The propagation equations for the non-local variables Υ_k and $\mathcal{Q}_k$ are redundant since these variables are determined by the constraint equations (VII.9) and (VII.10):

$$\frac{6}{\kappa^2}\frac{\rho}{\lambda}\Upsilon_k = 2\left(\frac{k}{a}\right)^2\Phi_k + 2H\left(\frac{k}{a}\right)(\mathcal{Z}_k - \sigma_k) - \kappa^2\rho\left(1+\frac{\rho}{\lambda}\right)\Delta_k + \frac{6}{\kappa^2}\frac{\rho}{\lambda}\mathcal{P}_k, \tag{VII.28}$$

$$\frac{3}{\kappa^2}\frac{\rho}{\lambda}\mathcal{Q}_k = \frac{1}{3}\left(\frac{k}{a}\right)^2(\mathcal{Z}_k - \sigma_k) - \frac{\kappa^2\rho}{2}\left(1+\frac{\rho}{\lambda}\right)q_k. \tag{VII.29}$$

Substituting these expressions in the right-hand sides of Eqs. (VII.2) and (VII.3) we find

$$\left(\frac{k}{a}\right)^2\left(\dot{\Phi}_k + H\Phi_k\right) + \frac{2\kappa^2\rho}{3}\left(\frac{k}{a}\right)\left(1+\frac{\rho}{\lambda}\right)\sigma_k - \frac{1}{3}\left(\frac{k}{a}\right)^3(\mathcal{Z}_k - \sigma_k) = \frac{3}{\kappa^2}\frac{\rho}{\lambda}(3H\mathcal{P}_k - \dot{\mathcal{P}}_k) \ , \tag{VII.30}$$

$$\left(\frac{k}{a}\right)\dot{\mathcal{Z}}_k + H\left(\frac{k}{a}\right)\mathcal{Z}_k + \kappa^2\rho\left(\frac{2\rho}{\lambda}\right)\Delta_k + 2\left(\frac{k}{a}\right)^2\Phi_k + 2H\left(\frac{k}{a}\right)(\mathcal{Z}_k - \sigma_k) = -\frac{6\rho}{\kappa^2\lambda}\mathcal{P}_k \ , \tag{VII.31}$$

$$\left(\frac{k}{a}\right)(\dot{\sigma}_k + H\sigma_k) + \left(\frac{k}{a}\right)^2\Phi_k = \frac{3}{\kappa^2}\frac{\rho}{\lambda}\mathcal{P}_k \ , \tag{VII.32}$$

$$\dot{\Delta}_k + \frac{k}{a}\left(\frac{4}{3}\mathcal{Z}_k - q_k\right) = 0 \ , \tag{VII.33}$$

$$\dot{q}_k + \frac{1}{3}\frac{k}{a}\Delta_k = 0 \ . \tag{VII.34}$$

These equations describe the evolution of the intrinsic perturbations to the brane. The usual general relativistic constraint equations are now replaced by the constraints (VII.28) and (VII.29) which determine two of the non-local variables. The lack of a propagation equation for $\mathcal{P}_k$ reflects the incompleteness of the 1+3 dimensional description of braneworld dynamics.

In the following it will prove convenient to adopt the dimensionless independent variable

$$x = \frac{k}{Ha} \ , \tag{VII.35}$$

which is (to within a factor of 2π) the ratio of the Hubble length to the wavelength of the perturbations. Using the (modified) Friedmann equations for the background in radiation domination, and with $\mathcal{U}=0$, we find that

$$\frac{dx}{dt} = \frac{k}{a}\left(\frac{2+3\rho/\lambda}{2+\rho/\lambda}\right). \tag{VII.36}$$

The relative importance of the local (quadratic) braneworld corrections to the Einstein equation depends on the dimensionless ratio ρ/λ. In the low-energy limit, $\rho \ll \lambda$, the quadratic local corrections can be neglected although the non-local corrections $\mathcal{E}_{ab}$ may still be important. In the opposite (high-energy) limit the quadratic corrections dominate over the terms that are linear in the energy-momentum tensor. We now consider these two limits separately.

A. Low-energy regime

In the low-energy regime we have $dx/dt \approx k/a$ and $x \approx k\tau$. The total energy density ρ is proportional to x^{-4}. Denoting derivatives with respect to x with a prime, using $\rho \ll \lambda$, and assuming that we can neglect the term involving $(\rho/\lambda)\Delta_k$ in Eq. (VII.31) compared to the other terms, we find

$$3x^2\Phi_k' + 3x\Phi_k + (6+x^2)\sigma_k - x^2\mathcal{Z}_k = \frac{27}{\kappa^4\lambda}(3\mathcal{P}_k - x\mathcal{P}_k') \tag{VII.37}$$

$$x^2\mathcal{Z}_k' + 3x\mathcal{Z}_k + 2x^2\Phi_k - 2x\sigma_k = -\frac{18}{\kappa^4\lambda}\mathcal{P}_k \tag{VII.38}$$

$$x^2\sigma_k' + x\sigma_k + x^2\Phi_k = \frac{9}{\kappa^4\lambda}\mathcal{P}_k \tag{VII.39}$$

$$\Delta_k' + \frac{4}{3}\mathcal{Z}_k - q_k = 0 \tag{VII.40}$$

$$q_k' + \frac{1}{3}\Delta_k = 0 \tag{VII.41}$$

Combining these equations we find an inhomogeneous, second-order equation for Φ_k:

$$3x\Phi_k'' + 12\Phi_k' + x\Phi_k = F_k(x), \tag{VII.42}$$

where

$$F_k(x) \equiv -\frac{27}{\kappa^4\lambda}\left[\mathcal{P}_k'' - \frac{\mathcal{P}_k'}{x} + \left(\frac{2}{x^3} - \frac{3}{x^2} + \frac{1}{x}\right)\mathcal{P}_k\right]. \tag{VII.43}$$

In general relativity the same second-order equation holds for Φ_k but with $F_k(x)=0$.

The presence of terms involving the non-local anisotropic stress on the right-hand side of Eq. (VII.42) ensure that Φ_k cannot be evolved on the brane alone. The resolution of this problem will require careful analysis of the bulk dynamics in five dimensions. In this paper our aims are less ambitious; we shall solve Eq. (VII.42) with $\mathcal{P}_k = 0$. Although we certainly do not expect $\mathcal{P}_{ab} = 0$[4], the solutions of the homogeneous equation may still prove a useful starting point for a more complete analysis. For example, they allow one to construct Green's functions for Eq. (VII.42) which could be used to assess the impact of specific ansatze for $\mathcal{P}_{ab}$ [33].

[4] We have not investigated the consistency of the condition $\mathcal{P}_{ab} = 0$ with the five-dimensional bulk dynamics in the presence of a perturbed brane.

With $\mathcal{P}_k = 0$ we can solve Eqs. (VII.37)–(VII.41) analytically to find

$$\Phi_k = \frac{c_1}{x^3}\left[3\sin\left(\frac{x}{\sqrt{3}}\right) - x\sqrt{3}\cos\left(\frac{x}{\sqrt{3}}\right)\right] + \frac{c_2}{x^3}\left[3\cos\left(\frac{x}{\sqrt{3}}\right) + x\sqrt{3}\sin\left(\frac{x}{\sqrt{3}}\right)\right], \tag{VII.44}$$

$$\sigma_k = \frac{3}{x^2}\left[c_2\cos\left(\frac{x}{\sqrt{3}}\right) + c_1\sin\left(\frac{x}{\sqrt{3}}\right)\right] + \frac{c_3}{x}, \tag{VII.45}$$

$$\mathcal{Z}_k = \frac{c_3(6+x^2)}{x^3} + \frac{6\sqrt{3}}{x^3}\left[c_1\cos\left(\frac{x}{\sqrt{3}}\right) - c_2\sin\left(\frac{x}{\sqrt{3}}\right)\right] + \frac{6}{x^2}\left[c_2\cos\left(\frac{x}{\sqrt{3}}\right) + c_1\sin\left(\frac{x}{\sqrt{3}}\right)\right], \tag{VII.46}$$

$$\begin{aligned}\Delta_k &= c_4\cos\left(\frac{x}{\sqrt{3}}\right) + c_5\sin\left(\frac{x}{\sqrt{3}}\right) + \frac{4c_3}{x^2} + \frac{4}{x}\left[c_2\cos\left(\frac{x}{\sqrt{3}}\right) + c_1\sin\left(\frac{x}{\sqrt{3}}\right)\right] \\ &+ \left(\frac{4\sqrt{3}}{x^2} - \frac{2}{\sqrt{3}}\right)\left[c_1\cos\left(\frac{x}{\sqrt{3}}\right) - c_2\sin\left(\frac{x}{\sqrt{3}}\right)\right],\end{aligned} \tag{VII.47}$$

$$\begin{aligned}q_k &= \frac{c_5}{\sqrt{3}}\cos\left(\frac{x}{\sqrt{3}}\right) - \frac{c_4}{\sqrt{3}}\sin\left(\frac{x}{\sqrt{3}}\right) + \frac{4c_3}{3x} + \frac{4x}{\sqrt{3}}\left[c_1\cos\left(\frac{x}{\sqrt{3}}\right) - c_2\sin\left(\frac{x}{\sqrt{3}}\right)\right] \\ &+ \frac{2}{3}\left[c_2\cos\left(\frac{x}{\sqrt{3}}\right) + c_1\sin\left(\frac{x}{\sqrt{3}}\right)\right].\end{aligned} \tag{VII.48}$$

The mode labelled by c_3 is the CDM velocity isocurvature mode discussed earlier. The modes labelled by c_1 and c_2 are the same as in general relativity; they describe the adiabatic growing and decaying solutions respectively. However, in the low-energy limit we also find two additional isocurvature modes (c_4 and c_5) that are not present in general relativity. These arise from the two additional degrees of freedom Υ_k and $\mathcal{Q}_k$ present in the braneworld model (with $P_k = 0$). The mode c_4 initially has non-zero but compensating gradients in the total matter and non-local densities, and c_5 initially has compensated energy fluxes. Formally these isocurvature solutions violate the assumption that the term involving $(\rho/\lambda)\Delta_k$ be negligible compared to the other terms in Eq. (VII.31) since all other terms vanish. In practice, there will be some gravitational back-reaction onto the other gauge-invariant variables controlled by the dimensionless quantity ρ/λ, but the general character of these isocurvature modes will be preserved for $\rho/\lambda \ll 1$.

B. High-energy regime

We now turn to the high-energy regime, where the quadratic terms in the stress-energy tensor dominate the (local) linear terms. In this limit the scale factor $a \propto t^{1/4}$. The modification to the expansion rate leads to an increase in the amplitude of scalar and tensor fluctuations produced during high-energy inflation [33]. With $\mathcal{U} = 0$ in the background, and $\rho \gg \lambda$, the Hubble parameter is approximately

$$H^2 \approx \frac{1}{36}\tilde{\kappa}^4\rho^2, \tag{VII.49}$$

and $dx/dt \approx 3k/a$. In terms of conformal time τ, $x \approx 3k\tau$.

1. *Power series solutions for the high-energy regime*

It is convenient to rescale the non-local variables by the dimensionless quantity $\kappa^4\rho$. Thus we define

$$\bar{\Upsilon}_k \equiv \frac{\Upsilon_k}{\kappa^4\rho}, \tag{VII.50}$$

$$\bar{\mathcal{Q}}_k \equiv \frac{\mathcal{Q}_k}{\kappa^4\rho}, \tag{VII.51}$$

$$\bar{\mathcal{P}}_k \equiv \frac{\mathcal{P}_k}{\kappa^4\rho}. \tag{VII.52}$$

The fractional total (effective) density perturbation and energy flux can be written in terms of the barred variables [e.g. $\bar{\Upsilon}_a \equiv \Upsilon_a/(\kappa^4\rho)$] in the high-energy limit as

$$\frac{aD_a\rho^{\rm tot}}{\rho^{\rm tot}} \approx 2(\Delta_a + 6\bar{\Upsilon}_a), \tag{VII.53}$$

$$q_a^{\rm tot} \approx \frac{2\rho^{\rm tot}}{\rho}(q_a + 6\bar{\mathcal{Q}}_a). \tag{VII.54}$$

Making the high-energy approximation $\rho \gg \lambda$ in Eqs. (VII.30)–(VII.34), we obtain

$$9x^2\Phi_k^{'} + 3x\Phi_k + (12+x^2)\sigma_k - x^2\mathcal{Z}_k = 54\left[\frac{7\bar{\mathcal{P}}_k}{x} - 3\bar{\mathcal{P}}_k^{\prime}\right] , \tag{VII.55}$$

$$3x^2\mathcal{Z}_k^{'} + 3x\mathcal{Z}_k - 2x\sigma_k + 2x^2\Phi_k + 12\Delta_k = -36\bar{\mathcal{P}}_k , \tag{VII.56}$$

$$3x\sigma_k^{'} + \sigma_k + x\Phi_k = 18\frac{\bar{\mathcal{P}}_k}{x} , \tag{VII.57}$$

$$\Delta_k^{'} - \frac{1}{3}q_k + \frac{4}{9}\mathcal{Z}_k = 0 , \tag{VII.58}$$

$$q_k^{'} + \frac{1}{9}\Delta_k = 0 . \tag{VII.59}$$

The non-local quantities $\bar{\Upsilon}_k$ and $\bar{\mathcal{Q}}_k$ are determined by the constraints

$$\bar{\Upsilon}_k = \frac{1}{18}x^2\Phi_k + \frac{1}{18}x(\mathcal{Z}_k - \sigma_k) - \frac{1}{6}\Delta_k + \bar{\mathcal{P}}_k, \tag{VII.60}$$

$$\bar{\mathcal{Q}}_k = \frac{1}{54}x^2(\mathcal{Z}_k - \sigma_k) - \frac{1}{6}q_k. \tag{VII.61}$$

We can manipulate Eqs. (VII.55)–(VII.59) to obtain a fourth-order equation for the gravitational potential Φ_k:

$$729x^2\frac{\partial^4\Phi_k}{\partial x^4} + 3888x\frac{\partial^3\Phi_k}{\partial x^3} + (1782+54x^2)\frac{\partial^2\Phi_k}{\partial x^2} + 144x\frac{\partial\Phi_k}{\partial x} + (90+x^2)\Phi_k = F_k(x) , \tag{VII.62}$$

where

$$F_k(x) = -\frac{54}{x^4}\left(243x^4\frac{\partial^4\bar{\mathcal{P}}_k}{\partial x^4} - 810x^3\frac{\partial^3\bar{\mathcal{P}}_k}{\partial x^3} + 18x^2(135+2x^2)\frac{\partial^2\bar{\mathcal{P}}_k}{\partial x^2} - 30x(162+x^2)\frac{\partial\bar{\mathcal{P}}_k}{\partial x} + [x^4 + 30(162+x^2)]\bar{\mathcal{P}}_k\right) . \tag{VII.63}$$

Since we do not have an evolution equation for $\bar{\mathcal{P}}_k$ we adopt the strategy taken in the low-energy limit and look for solutions of the homogeneous equations ($\bar{\mathcal{P}}_k = 0$). In principle one can use these solutions to construct formal solutions of the inhomogeneous equations with Green's method.

To solve Eq. (VII.62) with $\bar{\mathcal{P}}_k = 0$ we construct a power series solution for $\Phi_k(x)$:

$$\Phi_k(x) = x^m\sum_{n=0}^{\infty} a_n x^n , \tag{VII.64}$$

where $a_0 \neq 0$. The indicial equation for m is

$$m(m-1)(3m+5)(3m-4) = 0 . \tag{VII.65}$$

For each value of m we substitute into Eq. (VII.62) and solve the resulting recursion relations for the $\{a_n\}$. We then obtain the other gauge-invariant variables by direct integration. The original set of equations (VII.55)–(VII.59) has five degrees of freedom, so we expect one additional solution with $\Phi_k = 0$. This solution is the CDM isocurvature solution discussed earlier, and has a finite series expansion:

$$\Phi_k = 0 , \tag{VII.66}$$

$$\sigma_k = Cx^{-\frac{1}{3}} , \tag{VII.67}$$

$$\mathcal{Z}_k = Cx^{-\frac{7}{3}}(12+x^2) , \tag{VII.68}$$

$$\Delta_k = 4Cx^{-\frac{4}{3}} , \tag{VII.69}$$

$$q_k = \frac{4}{3}Cx^{-\frac{1}{3}} , \tag{VII.70}$$

where C is a constant. The non-local variables vanish.

The first two terms of the mode with $m = 0$ are

$$\Phi_k = b_1 \left(1 - \frac{5}{198} x^2\right) , \tag{VII.71}$$

$$\sigma_k = b_1 \left(-\frac{1}{4} x + \frac{1}{396} x^3\right) , \tag{VII.72}$$

$$\mathcal{Z}_k = b_1 \left(-\frac{3}{4} x + \frac{5}{864} x^3\right) , \tag{VII.73}$$

$$\Delta_k = b_1 \left(\frac{1}{6} x^2 - \frac{1}{864} x^4\right) , \tag{VII.74}$$

$$q_k = b_0 \left(-\frac{1}{162} x^3 + \frac{1}{38880} x^5\right) , \tag{VII.75}$$

$$\bar{\Upsilon}_k = b_1 \left(-\frac{1}{972} x^4 + \frac{1}{249480} x^6\right) , \tag{VII.76}$$

$$\bar{\mathcal{Q}}_k = b_1 \left(-\frac{2}{243} x^3 + \frac{1}{17820} x^5\right) , \tag{VII.77}$$

where b_1 is a constant. The form of this solution is similar to the adiabatic growing mode of general relativity.

The mode corresponding to $m = 1$ is

$$\Phi_k = b_2 \left(x - \frac{13}{1890} x^3\right) , \tag{VII.78}$$

$$\sigma_k = b_2 \left(-\frac{1}{7} x^2 + \frac{1}{1890} x^4\right) , \tag{VII.79}$$

$$\mathcal{Z}_k = b_2 \left(\frac{72}{7} - \frac{12}{35} x^2\right) , \tag{VII.80}$$

$$\Delta_k = b_2 \left(-\frac{18}{7} x + \frac{1}{15} x^3\right) , \tag{VII.81}$$

$$q_k = b_2 \left(6 + \frac{1}{7} x^2\right) , \tag{VII.82}$$

$$\bar{\Upsilon}_k = b_2 \left(x + \frac{1}{30} x^3\right) , \tag{VII.83}$$

$$\bar{\mathcal{Q}}_k = b_2 \left(-1 + \frac{1}{6} x^2\right) , \tag{VII.84}$$

with b_2 a constant. As $t \to 0$ there are non-zero but compensating contributions to the effective peculiar velocity $q_a^{\rm tot}/\rho^{\rm tot}$ from the matter and the non-local energy fluxes. The contributions of these components to the fractional total density perturbation $aD_a\rho^{\rm tot}/\rho^{\rm tot}$ vanish as $t \to 0$. It follows that this solution describes an isocurvature velocity mode where the early time matter and non-local (Weyl) components have equal but opposite peculiar velocities in the CDM frame. The existence of such isocurvature modes was predicted in Refs [11] and [21] for large-scale density perturbations.

The mode corresponding to $m = -\frac{5}{3}$ is singular as $t \to 0$ (it is a decaying mode):

$$\Phi_k = b_3 x^{-\frac{5}{3}} \left(1 - \frac{5}{18}x^2\right) , \tag{VII.85}$$

$$\sigma_k = b_3 x^{-\frac{2}{3}} \left(1 + \frac{1}{18}x^2\right) , \tag{VII.86}$$

$$\mathcal{Z}_k = b_3 \left(\frac{14}{99}x^{\frac{4}{3}} - \frac{1217}{1590435}x^{\frac{10}{3}}\right) , \tag{VII.87}$$

$$\Delta_k = b_3 \left(-\frac{8}{297}x^{\frac{7}{3}} + \frac{64}{433755}x^{\frac{13}{3}}\right) , \tag{VII.88}$$

$$q_k = b_3 \left(\frac{4}{4455}x^{\frac{10}{3}} - \frac{4}{1301265}x^{\frac{16}{3}}\right) , \tag{VII.89}$$

$$\bar{\Upsilon}_k = b_3 \left(-\frac{1}{162}x^{\frac{7}{3}} + \frac{7}{43740}x^{\frac{13}{3}}\right) , \tag{VII.90}$$

$$\bar{\mathcal{Q}}_k = b_3 \left(-\frac{1}{54}x^{\frac{4}{3}} + \frac{7}{4860}x^{\frac{10}{3}}\right) . \tag{VII.91}$$

A similar mode is found in general relativity but there the decay of Φ_k is more rapid ($\Phi_k \propto x^{-3}$) on large scales.
Finally, for $m = \frac{4}{3}$ we have

$$\Phi_k = b_4 x^{\frac{4}{3}} \left(1 - \frac{17}{3150}x^2\right) , \tag{VII.92}$$

$$\sigma_k = b_4 x^{\frac{4}{3}} \left(-\frac{1}{8}x + \frac{17}{44100}x^3\right) , \tag{VII.93}$$

$$\mathcal{Z}_k = b_4 x^{\frac{1}{3}} \left(\frac{27}{2} - \frac{117}{392}x^2\right) , \tag{VII.94}$$

$$\Delta_k = b_4 x^{\frac{4}{3}} \left(-\frac{9}{2} + \frac{3}{49}x^2\right) , \tag{VII.95}$$

$$q_k = b_4 x^{\frac{4}{3}} \left(\frac{3}{14}x - \frac{1}{637}x^3\right) , \tag{VII.96}$$

$$\bar{\Upsilon}(x) = b_4 x^{\frac{4}{3}} \left(\frac{3}{2} + \frac{1}{28}x^2\right) , \tag{VII.97}$$

$$\bar{\mathcal{Q}}(x) = b_4 x^{\frac{4}{3}} \left(\frac{3}{14}x - \frac{29}{9828}x^3\right) . \tag{VII.98}$$

In this mode the universe asymptotes to an FRW (brane) model in the past as $t \to 0$. Note that this requires careful cancellation between $aD_a\rho^{\text{tot}}/\rho^{\text{tot}}$ and q_a^{tot} to avoid a singularity in the gravitational potential Φ_k (which would diverge as $x^{-2/3}$ without such cancellation). Like the velocity isocurvature mode ($m = 1$) discussed above, this mode has no analogue in general relativity.

VIII. A COVARIANT EXPRESSION FOR THE TEMPERATURE ANISOTROPY

In this section we discuss the line of sight solution to the Boltzmann equation for the scalar contribution to the gauge-invariant temperature anisotropy $\delta_T(e)$ of the CMB in braneworld models. We employ the 1+3 covariant approach, and show that our result is equivalent to that given recently by Langlois et al [11] using the Bardeen formalism.

Over the epoch of interest the individual matter constituents of the universe interact with each other under gravity only, except for the photons and baryons (including the electrons), which are also coupled through Thomson scattering. The variation of the gauge-invariant temperature perturbation $\delta_T(e)$, where e^a is the (projected) photon propagation direction, along the line of sight is given by the (linearized) covariant Boltzmann equation (valid for scalar, vector,

and tensor modes) [34]:

$$\begin{aligned}\delta_T(e)' + \sigma_T n_e \delta_T(e) = -\sigma_{ab} e^a e^b - A_a e^a - \frac{e^a D_a \rho^{(\gamma)}}{4\rho^{(\gamma)}} - \frac{D^a q_a^{(\gamma)}}{4\rho^{(\gamma)}} \\ + \sigma_T n_e \left(v_a^{(b)} e^a + \frac{3}{16}\rho^{(\gamma)} \pi_{ab}^{(\gamma)} e^a e^b \right) ,\end{aligned} \tag{VIII.1}$$

where the prime denotes the derivative with respect to a parameter λ defined along the line of sight by $d\lambda = -u_a dx^a$.

Following the steps in Ref. [34], we expand the right-hand side of Eq. (VIII.1) in scalar harmonics and integrate along the line of sight from the early universe to the observation point R. Neglecting effects due to the finite thickness of the last scattering surface, on integrating by parts we find that the temperature anisotropy involves the quantity

$$\left(\frac{a}{k}\sigma_k'\right)' + \frac{1}{3}\frac{k}{a}(\sigma_k - \mathcal{Z}_k) + A_k' - HA_k = -2\dot{\Phi}_k + \left(\frac{a}{k}\right)^2 I \tag{VIII.2}$$

integrated along the line of sight (after multiplying with $Q^{(k)}$). In simplifying Eq. (VIII.2) we have made use of the derivative of the shear propagation equation (V.9), substituted for q_k and $\mathcal{Z}_k$ from equations (V.8) and (V.10), and finally used equations (III.15) and (IV.13). The quantity I is the total sum of all the braneworld corrections:

$$I = \left(\frac{a}{k}\right)^2 \left[\dot{I}_1 + \frac{1}{3}\Theta I_1 + I_2 + \frac{1}{3}\left(\frac{k}{a}\sigma_k\right) I_3 + \frac{1}{2}\left(\frac{k}{a}\right) I_4 \right] , \tag{VIII.3}$$

where

$$\begin{aligned} I_1 &= \frac{1}{24}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 [-(3\gamma-2)\kappa^4\rho^2\pi_k + 12\rho\mathcal{P}_k] , \\ I_2 &= \frac{1}{72}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \Bigg\{ -\kappa^4 \Bigg[6\left(\frac{k}{a}\right)\gamma\rho^2\sigma_k - 3(\dot{\rho}+3\dot{P})\rho\pi_k - 3(3\gamma-2)\rho(\rho\dot{\pi}_k + \dot{\rho}\pi_k) - 6\left(\frac{k}{a}\right)\rho^2 q_k \\ &\quad - (3\gamma-2)\rho^2\Theta\pi_k \Bigg] - 48\left(\frac{k}{a}\right)\mathcal{U}\sigma_k - 36(\dot{\rho}\mathcal{P}_k + \rho\dot{\mathcal{P}}_k) + 36\left(\frac{k}{a}\right)\rho\mathcal{Q}_k - 12\rho\Theta\mathcal{P}_k \Bigg\} , \\ I_3 &= \frac{1}{12}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 [(3\gamma-1)\kappa^4\rho^2 + 12\mathcal{U}] , \\ I_4 &= \frac{1}{18}\sigma_k\tilde{\kappa}^4\rho^2 + \frac{2}{3}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \mathcal{U}\sigma_k - \frac{1}{24}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 (4\kappa^4\rho^2 q_k + 24\rho\mathcal{Q}_k) . \end{aligned} \tag{VIII.4}$$

A lengthy calculation making use of the propagation and constraint equations shows that $I = 0$. The final result for the temperature anisotropies is then

$$\begin{aligned}[\delta_T(e)]_R = -\sum_k \left[\left(\frac{1}{4}\Delta_k^{(\gamma)} + \frac{a}{k}\dot{\sigma}_k + A_k \right) Q^{(k)} \right]_A + \sum_k [(v_k^{(b)} - \sigma_k) e^a Q_a^{(k)}]_A \\ + \frac{3}{16}\sum_k (\pi_k^{(\gamma)} e^a e^b Q_{ab}^{(k)})_A + 2\sum_k \int_{\lambda_A}^{\lambda_R} \dot{\Phi}_k Q^{(k)} d\lambda ,\end{aligned} \tag{VIII.5}$$

where the event A is the intersection of the null geodesic with the last scattering surface.

In retrospect, one could re-derive the result for the temperature anisotropy in braneworld models much more simply by retaining the effective stress-energy variables ρ^{tot}, P^{tot}, q_a^{tot} and π_{ab}^{tot} in the propagation and constraint equations used in the manipulation of the left-hand side of Eq. (VIII.2), rather than isolating the braneworld contributions.

If we adopt the longitudinal gauge, defined by $\sigma_{ab} = 0$, we find that the electric part of the Weyl tensor and the acceleration are related by $\Phi_k = -A_k$ if the total anisotropic stress π_{ab}^{tot} vanishes. It follows that in this zero shear frame we recover the result found by Langlois et al [11].

Regarding the imprint of braneworld effects on the CMB, we note several possible sources. Once the universe enters the low-energy regime the dynamics of the perturbations are essentially general relativistic in the absence of non-local anisotropic stress (see Sec. VII A). If $\mathcal{P}_{ab}$ really were zero, the only imprints of the braneworld on the CMB could arise from modifications to the power spectrum (and cross correlations) between the various low-energy modes. Since there are two additional isocurvature modes in the low-energy universe due to braneworld effects, it need not be the case that adiabatic fluctuations produced during high-energy (single-field) inflation give rise to a low-energy

universe dominated by the growing, adiabatic, general-relativistic mode. The possibility of exciting the low-energy isocurvature (brane) modes from plausible fluctuations in the high-energy regime is worthy of further investigation. In practice we do not expect $\mathcal{P}_{ab} = 0$. In this case the non-local anisotropic stress provides additional driving terms to the dynamics of the fluctuations, and we can expect a significant manifestation of five-dimensional Kaluza-Klein effects on the CMB anisotropies.

IX. CONCLUSION

In this paper we have discussed the dynamics of cosmological scalar perturbations in the braneworld scenario from the viewpoint of brane-bound observers making use of the 1+3 covariant approach. We only considered matter components present in the ΛCDM model, but it is straightforward to include other components such as hot dark matter.

We presented approximate, analytic solutions for the fluctuations in the low-energy universe under the assumption that the non-local anisotropic stress was negligible. We obtained two additional isocurvature modes not present in general relativity in which the additional density gradients or peculiar velocities of the total matter are compensated by fluctuations in the non-local variables. In practice we do not expect the non-local anisotropic stress $\mathcal{P}_{ab}$ necessarily to be negligible; in this case our solutions to the homogeneous equations should form a useful starting point for the construction of solutions to the driven equations. By adopting a four-dimensional approach our presentation is necessarily limited. In particular, we cannot predict the evolution of $\mathcal{P}_{ab}$ on the brane. However, the four-dimensional approach should be well-suited to a phenomenological description of these five dimensional Kaluza-Klein modes. A simple possibility is to adopt an ansatz for the evolution of $\mathcal{P}_{ab}$ [33], and this will be explored further in a future paper.

We also presented solutions to the perturbation equations in the high-energy regime where braneworld effects dominate. In this limit the gravitational potential satisfies a fourth-order differential equation which we were unable to solve analytically (even with $\mathcal{P}_{ab} = 0$). Instead we constructed power series solutions for the case where the non-local anisotropic stresses vanish; these should prove useful for setting initial conditions in the high-energy regime when performing a numerical solution of the perturbation equations. We found two additional modes over those present in general relativity, one of which can be described as a (brane) isocurvature velocity mode. We also showed that the adiabatic decaying mode varies less rapidly than in general relativity on large scales.

The detailed calculation of braneworld imprints on the CMB (in the phenomenological approach discussed above) will be described in a future paper. Here we showed with the 1+3 covariant approach that the line of sight integral for the CMB temperature anisotropies is unchanged in form from general relativity. We also noted that excitation of the additional isocurvature modes present in the low-energy universe could provide an additional imprint in the CMB, over and above that due to the non-local anisotropic stress.

Acknowledgments

B. L., P. D., and A. L. thank the organisers of the 8th course of the International School of Astrophysics "D. Chalonge" held in Erice, 2000 at which this work was initiated. B. L. thanks the Relativity and Cosmology Group, University of Portsmouth for hospitality, and R. Maartens for useful correspondences over issues connected with the non-local anisotropic stress, and for helpful comments on an earlier draft of the paper. B. L. also thanks B. Bassett, C. Gordon, J. C. Hwang, D. Langlois, A. Lewis, Y. L. Loh, C. J. A. P. Martins, J. Soda, D. Wands and C. Van de Bruck for insightful comments. B. L. is supported by an Overseas Research Studentship, the Cambridge Commonwealth Trust and the Lee Foundation, Singapore. P. D. thanks the NRF (South Africa) for financial support and the Cavendish Laboratory for hospitality. A. C. acknowledges a PPARC Postdoctoral Fellowship.

APPENDIX A: ENERGY FRAME EQUATIONS IN THE RADIATION-DOMINATED ERA

In this appendix we present a complete set of evolution equations for the total matter variables in the matter energy frame, $q_a = 0$. Note that the four-velocity of the energy frame is not necessarily a timelike eigenvector of the Einstein tensor in the presence of the non-local braneworld corrections to the effective stress-energy tensor. We assume that the matter is radiation dominated, the non-local energy density vanishes in the background, and we ignore local anisotropic stresses. We also assume that the baryons and CDM make a negligible contribution to the fractional gradient in the total matter energy density and to the energy flux, thus excluding the CDM and baryon

isocurvature modes. We also give the evolution equations for the non-local density gradient and energy flux in the matter energy frame.

Denoting the variables in the energy frame by an overbar[5], the relevant equations for scalar perturbations are

$$\dot{\bar{\Delta}}_a = \frac{1}{3}\Theta\bar{\Delta}_a - \frac{4}{3}\bar{\mathcal{Z}}_a\,, \tag{A.A1}$$

$$\dot{\bar{\mathcal{Z}}}_a = -\frac{2}{3}\Theta\bar{\mathcal{Z}}_a - \frac{1}{4}D^2\bar{\Delta}_a - \left(\frac{\tilde{\kappa}}{\kappa}\right)^4\rho\bar{\Upsilon}_a - \frac{1}{2}\kappa^2\rho\bar{\Delta}_a\left(1+\frac{5\rho}{\lambda}\right)\,, \tag{A.A2}$$

$$\dot{\bar{\Upsilon}}_a = -\frac{a}{\rho}D^2\bar{\mathcal{Q}}_a\,, \tag{A.A3}$$

$$\dot{\bar{\mathcal{Q}}}_a = -\frac{4}{3}\Theta\bar{\mathcal{Q}}_a - \frac{\rho}{3a}\bar{\Upsilon}_a - \frac{2\kappa^4\rho^2}{9a}\bar{\Delta}_a - D^b\bar{\mathcal{P}}_{ab}. \tag{A.A4}$$

Solutions of these equations are related to those in the CDM frame (Sec. VII) by linearising the frame transformations given in Ref. [35]. If the CDM projected velocity is $\bar{v}_a^{(c)}$ in the energy frame, the variables in the CDM frame are given by

$$\Delta_a = \bar{\Delta}_a - \frac{4}{3}a\Theta\bar{v}_a^{(c)}\,, \tag{A.A5}$$

$$\mathcal{Z}_a = \bar{\mathcal{Z}}_a + \frac{1}{a}D_aD^b\bar{v}_b^{(c)} - \frac{2\kappa^2\rho}{a}\left(1+\frac{\rho}{\lambda}\right)\bar{v}_a^{(c)}\,, \tag{A.A6}$$

$$\Upsilon_a = \bar{\Upsilon}_a\,, \tag{A.A7}$$

$$\mathcal{Q}_a = \bar{\mathcal{Q}}_a\,, \tag{A.A8}$$

$$q_a = -\frac{4}{3}\rho\bar{v}_a^{(c)}\,, \tag{A.A9}$$

where we have used $\mathcal{U}=0$ in the background. The CDM peculiar velocity evolves in the energy frame according to

$$\dot{\bar{v}}_a^{(c)} = -\frac{1}{3}\Theta\bar{v}_a^{(c)} + \frac{1}{4a}\bar{\Delta}_a. \tag{A.A10}$$

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[2] R. Maartens, D. Wands, B. Bassett, and I. Heard, Phys. Rev. D **62**, 041301 (2000).
[3] P. de Bernardis et al , Nature **404**, 955 (2000).
[4] S. Hanany et al , Astrophys. J. Lett. **545**, L5 (2000).
[5] L. Anchordoqui C. Nunez K. Olsen , JHEP **0010**, 050 (2000).
[6] H. A. Bridgman, K. Malik, and D. Wands, astro-ph/0107245 (2001).
[7] E. J. Copeland, A. Liddle, and J. Lidsey, Phys. Rev. D **64**, 023509 (2001).
[8] N. Deruelle, T. Dolezel, and J. Katz, Phys. Rev. D **63**, 083513 (2001).
[9] H. Kodama, A. Ishibashi,and O. Seta, Phys. Rev. D **62**, 064022 (2000).
[10] K. Koyama and J. Soda, hep-th/0108003 (2001).
[11] D. Langlois, R. Maartens, M. Sasaki, and D. Wands, Phys. Rev. D **63**, 084009 (2001).
[12] D. Langlois, R. Maartens, and D. Wands, Phys. Lett. B **489**, 259 (2000).
[13] D. Langlois, Phys. Rev. D **62**, 126012 (2000).
[14] D. Langlois, Phys. Rev. Lett. **86**, 2212 (2000).
[15] S. Mukohyama, Phys. Rev. D **62**, 084015 (2000).
[16] C. van de Bruck, M. Dorca, R. Brandenberger, and A. Lukas, Phys. Rev. D **62**, 123515 (2000).
[17] C. van de Bruck and M. Dorca, hep-th/0012073 (2000).
[18] C. van de Bruck, M. Dorca, C. Martins, and M. Parry, Phys. Lett. B. **495**, 183 (2000).
[19] D. Gorbunov, V. Rubakov, and S. Sibiryakov, hep-th/0108017 (2001).
[20] H. Kodama, hep-th/0012132 (2000).
[21] C. Gordon and R. Maartens, Phys. Rev. D **63**, 044022 (2001).

[5] This notation should not be confused with our use of the overbar to denote rescaling by $\kappa^4\rho$ in Sec. VII B 1.

[22] R. Maartens, Phys. Rev. D, **62**, 084023 (2000).
[23] R. Maartens, gr-qc/0101059 (2001).
[24] G. Ellis and M. Bruni, Phys. Rev. D **40**, 1804 (1989).
[25] B. Leong, A. Lewis, A. Challinor, P. Dunsby, A. Lasenby and R. Maartens (in preparation) .
[26] T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).
[27] A. Lewis, A. Challinor, and A. Lasenby, Astrophys. J. **538**, 473 (2000).
[28] P. Dunsby, M. Bruni, and G. Ellis, Astrophys. J. **395**, 34 (1992).
[29] A. Challinor, Gen. Rel. Grav. **32**, 1059 (2000).
[30] A. Challinor and A. Lasenby, Astrophys. J. **513**, 1 (1999).
[31] C. Ma and E. Bertschinger, Astrophys. J. **455**, 7 (1995).
[32] M. Bucher, K. Moodley, and N. Turok, Phys. Rev. D **62**, 083508 (2000).
[33] J. Barrow and R. Maartens, gr-qc/0108073 (2001).
[34] A. Challinor and A. Lasenby, Phys. Rev. D **58**, 023001 (1998).
[35] R. Maartens, T. Gebbie, and G. Ellis, Phys. Rev. D **59**, 083506 (1999).